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FORCED VIBRATIONS OF CONTINUOUS BEAMS

By Edward Saibel and Elio D'Appolonia

ENGINEERING MECHANICS DIVISION

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

FORCED VIBRATIONS OF CONTINUOUS BEAMS

By Edward Saibel[1] and Elio D'Appolonia[2]

Synopsis

The determination of the forced oscillations of a continuous beam due to an exciting force is presented in this paper. The procedure requires only a knowledge of the eigenfunctions and eigenvalues of the continuous beam from which all intermediate points of support have been removed (a simple beam). For the simple beam, these functions are known or can be readily evaluated.

The deflected shape of the continuous beam is developed in terms of the eigenfunctions of the simple beam.

The equations of motion are set up using the Lagrangian equations of motion, the intermediate supports being introduced through undetermined multipliers. The solution is found for any number of supports and a numerical example is given.

Introduction

The problem of the forced vibrations of beams has been a subject of discussion for over a century. The studies, in practically all cases, have been restricted to simple span structures. In the extension to continuous beams, the classical approach has been to cut the beam at the intermediate supports, destroying the continuity, and leaving a series of individual beams for each of which the solution can be found. Re-establishing the continuity of the beam by equating deflections, slopes, and moments at the sections that had been previously cut leads to a system of linear equations. In the case of the free oscillations of a continuous beam, the system of equations is homogeneous. The necessary condition for a nontrivial solution leads to the frequency equation from which the natural frequencies may be found. When the motion of the continuous beam is forced, the system of equations is nonhomogeneous

Note.—Written comments are invited for publication; the last discussion should be submitted by May 1, 1952.

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and may be solved, in theory, for the deflection of the beam as a function of time. In practice, however, the system of equations that arises is intractable, and as a result, if the solution is to be of practical use, some other method of analysis must be devised.

A solution for the special case of a beam of two equal spans and simply supported at its extremities has been developed,[3] but this solution is apparently not capable of complete generalization. It is the purpose of this paper to present a solution for the forced vibrations of all types of continuous beams subjected to a fixed pulsating load. The method used is an extension of the one developed by the senior author[4] for the solution of the natural frequencies of continuous beams. This method had been developed earlier, in connection with other problems, by A. Weinstein.[5]

Basically the steps taken in the solution are: (*a*) the intermediate supports are removed leaving an ordinary simple beam for which the vibration frequencies and modes (eigenvalues and eigenfunctions) are known. The term simple beam as used herein refers to a beam without intermediate supports, but one whose end conditions can be any combination of simple, fixed, or free; (*b*) the deflection of the beam at any point along its length is represented as a generalized Fourier series in terms of the eigenfunctions of the simple beam; (*c*) the constraints at the intermediate supports are introduced through undetermined multipliers; and (*d*) from the Lagrange equations of motion and the conditions of constraint at intermediate supports (such as no deflection at a support if that happens to be the case) equations are developed that yield the solution of the problem.

This procedure may be applied to beams of any number of spans of arbitrary length and of varying flexural rigidity, and with any type of conditions at the extreme ends of the beam.

The important thing is that the eigenfunctions for the simple beam (the beam that is left when the intermediate supports are removed) must be known or be obtainable. In many practical cases these eigenfunctions are known.

In this paper the theory is applied to the case of a two-span uniform beam of unequal lengths subjected to a fixed pulsating force.

Theory

Steady Vibrations.—The usual assumptions made in the classical theory of beam vibrations are assumed in this paper. Damping effects are neglected. The differential equation of the vibrating beam is

$$\frac{\partial^2}{\partial x^2}\left[E\,I\,(x)\,\frac{\partial^2 y}{\partial x^2}\right] + m\,(x)\,\frac{\partial^2 y}{\partial t^2} = P\,(x, t) \qquad (1)$$

in which E is the modulus of elasticity, I is the moment of inertia, x is the distance, $E\,I\,(x)$ is the flexural rigidity, $n(x)$ is the mass per unit length,

[3] "Transverse Vibration of a Two-Span Beam Under Action of a Moving Constant Force," by R. S. Ayre, George Ford, and L. S. Jacobsen, *Journal of Applied Mechanics*, American Society of Mechanical Engineers, Vol. 71, 1949, p. 683.

[4] "Vibration Frequencies of Continuous Beams," by Edward Saibel, *Journal of the Aeronautical Sciences*, Vol. 11, 1944, pp. 88–90.

[5] "Etude des spectres des équations aux derivées partielles de la théorie des plaques élastiques," by A. Weinstein, *Fascicule 88*, Mémorial des Sciences Mathématiques, Gauthier-Villars, Paris, 1937.

$y\ (x,\ t)$ is the deflection, and $P(x, t)$ is the externally applied force that, in this treatment, is exclusive of the intermediate or end supporting forces or moments.

As pointed out, the solution will be developed in the form

$$y = \sum_{r=1}^{\infty} q_r(t)\ \phi_r(x) \qquad (2)$$

in which the values of q_r are the generalized coordinates of the vibrating system and the ϕ_r-values are the eigenfunctions of the simple beam. The ϕ_r-values constitute a complete orthonormal set, that is

$$\int_o^L m(x)\ \phi_r(x)\ \phi_s(x)\ dx = \left.\begin{matrix} 0, & \text{when } r \neq s \\ 1, & \text{when } r = s \end{matrix}\right\}$$

in which L is the total span length. Furthermore, associated with each ϕ_r, it is assumed that there is only one eigenvalue $p_r = \sqrt{\lambda_r}$ that is the circular frequency in the rth mode of vibration.

To establish the amplitudes of the forced vibrations of the continuous beam, it is first necessary to determine the equations expressing the generalized coordinates as a function of the eigenfunctions and eigenvalues of the simple beam and the frequency of the applied load.

This is accomplished by the use of the Lagrange equations of motion, in which are introduced as many undetermined multipliers μ_i as there are intermediate supports. It is through these undetermined multipliers that the constraints on the beam at the intermediate supports are taken care of. In the case in which the beam is allowed no deflection at the point, $x = c_i$, this constraint (f_i) takes the form

$$f_i \equiv \sum_{r=1}^{r=\infty} q_r\ \phi_r\ (c_i) = 0 \qquad (3)$$

and assuming n such points along the beam, that is, n intermediate supports or a continuous beam of $(m + 1)$ spans, the Lagrange equation takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} - Q_r - \sum_{i=1}^{i=m} \mu_i(t)\ \frac{\partial f_i}{\partial q_r} = 0 \qquad (4)$$

In this equation, V is the potential energy due to bending, T is the kinetic energy, Q_r is the generalized force, and q'_r is $\frac{dq_r}{dt}$.

It can be shown that the potential energy due to bending is

$$V = \frac{1}{2}\int_o^L E\ I(x)\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \qquad (5)$$

It follows, on the substitution of Eq. 2 into Eq. 5 that

$$V = \frac{1}{2}\sum_{r=1}^{r=\infty}\int_o^L q^2_r\left[\frac{d^2}{dx^2}\left(E\ I(x)\ \frac{d^2\phi_r}{dx^2}\right)\right]\phi_r\ dx \qquad (6)$$

Furthermore, the eigenfunctions of the simple beam satisfy the equation

$$\frac{d^2}{dx^2}\left(E\,I(x)\,\frac{d^2\,\phi_r}{dx^2}\right) - m(x)\,p^2{}_r\,\phi_r = 0 \qquad (7)$$

and since the values of ϕ_r satisfy Eq. 3, the potential energy of the system becomes

$$V = \frac{1}{2}\sum_{r=1}^{r=\infty} p_r{}^2\,q^2{}_r \qquad (8)$$

in which p_r is the rth eigenvalue of the simple beam ($= \lambda_r$). The kinetic energy of a vibrating beam is given by

$$T = \frac{1}{2}\int_o^L m(x)\left(\frac{\partial y}{\partial t}\right)^2 dx \qquad (9)$$

Using Eq. 2 and Eq. 3 it can be shown that Eq. 9 takes the form

$$T = \frac{1}{2}\sum_{r=1}^{r=\infty} (q'_r)^2 \qquad (10)$$

In this work, the deflection of the beam at each intermediate support is zero, thus the constraint condition at the ith support is

$$f_i = \sum_{r=1}^{r=\infty} q_r\,\phi_r\,(c_i) = 0 \qquad (11)$$

defining $i = 1, 2, \ldots n$.

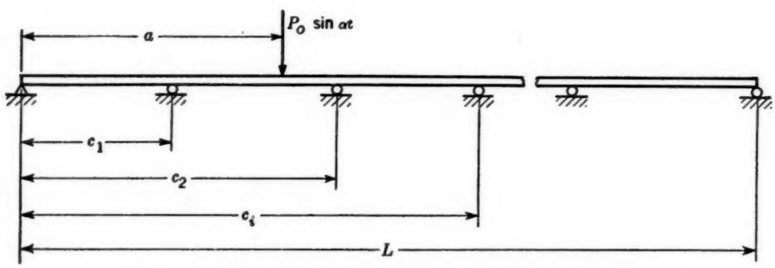

FIG. 1.—CONTINUOUS BEAM SUBJECT TO PULSATING LOAD

Substituting Eqs. 8, 10, and 11 into Lagrange's equations of motion (Eq. 4), the resulting differential equation to be satisfied by each generalized coordinate becomes

$$q''_r + p^2{}_r\,q_r = Q_r + \sum_{i=1}^{i=m} \mu_i\,\phi_r\,(c_i) \qquad (12)$$

with $r = 1, 2, \cdots \infty$ in which $q''_r = \dfrac{d^2q_r}{dt^2}$. The solution of Eq. 12 may be obtained in the usual manner, that is, as the sum of the particular and of the homogeneous solutions. The form of the particular solution will depend on the right hand side of the equation.

Consider the case of a continuous beam on which a load fixed in position $x = a$ but pulsating with circular frequency α is acting (Fig. 1). This load may be represented by $P_o \sin \alpha t$. The generalized force is

$$Q_r = (P_o \sin \alpha t)\, \phi_r(a) \qquad (13)$$

The particular solution may be formed by assuming

$$\mu_i = A_i \sin \alpha t \qquad (14a)$$

and

$$q_r = B_r \sin \alpha t \qquad (14b)$$

in which A_i and B_r are coefficients to be determined. After dividing out the factor $\sin \alpha t$, Eq. 12 takes the form

$$- B_r \alpha^2 + B_r p^2_r = P_o \phi_r(a) + \sum_{i=1}^{i=m} A_i \phi_r(c_i) \qquad (15)$$

from which

$$B_r = \frac{P_o \phi_r(a) + \sum_{i=1}^{i=m} A_i \phi_r(c_i)}{p^2_r - \alpha^2} \qquad (16)$$

The coefficient A_i may be determined from the constraint conditions

$$\sum_{r=1}^{r=\infty} B_r \phi_r(c_i) = 0 \qquad (17)$$

in which $i = 1, 2, \ldots m$.

On substituting Eq. 16 into Eq. 17

$$\sum_{r=1}^{\infty} \frac{P_o \phi_r(a)\, \phi_r(c_i) + \sum_{j=1}^{j=m} A_j \phi_r(c_j)\, \phi_r(c_i)}{p^2_r - \alpha^2} = 0 \qquad (18)$$

In expanded form, Eq. 18 becomes

$$A_1 \sum_{r=1}^{r=\infty} \frac{\phi^2_r(c_1)}{p^2_r - \alpha^2} + A_2 \sum_{r=1}^{r=\infty} \frac{\phi_r(c_1)\, \phi_r(c_2)}{p^2_r - \alpha^2} + \cdots = - P_o \sum_{r=1}^{r=\infty} \frac{\phi_r(a)\, \phi_r(c_1)}{p^2_r - \alpha^2} \qquad (19a)$$

$$A_1 \sum_{r=1}^{r=\infty} \frac{\phi_r(c_1)\, \phi_r(c_2)}{p^2_r - \alpha^2} + A_2 \sum_{r=1}^{r=\infty} \frac{\phi^2_r(c_2)}{p^2_r - \alpha^2} + \cdots = - P_o \sum_{r=1}^{r=\infty} \frac{\phi_r(a)\, \phi_r(c_2)}{p^2_r - \alpha^2} \qquad (19b)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

The solution of this set of simultaneous linear equations for the values of A can be obtained for any number of supports. Only the first few terms of the series need be considered as the series converges rapidly. Hence the steady state part of the forced vibration solution can be expressed by

$$(y)_p = \sum_{r=1}^{r=\infty} \frac{P_o \phi_r(a) + \sum_{i=1}^{i=m} A_i \phi_r(c_i)}{p^2_r - \alpha^2} \sin \alpha t\, \phi_r(x) \qquad (20)$$

Eq. 20 is valid provided $p_r \neq \alpha$.

When the forcing frequency is equal to one of the natural frequencies of the simple beam a slight modification must be made. It is apparent from Eq. 15 that when $p_r = \alpha$,

$$P_o\,\phi_r(a) + \sum_{i=1}^{i=m} A_i\,\phi_r(c_i) = 0 \quad (21)$$

consequently Eq. 16 becomes indeterminate, and B_r cannot be determined using it. However, the other values of B may still be found from this equation. The constraint Eq. 10 may be written

$$y(c, t) = \sum_{k=1}^{k=\infty *} q_k\,\phi_k(c) + q_r\,\phi_r(c) = 0 \quad (22)$$

in which $\sum_{k=1}^{k=\infty *}$ indicates the sum over k with the $(k = r)$ term omitted. Since this must hold for all values of the time it may be expressed

$$\sum_{k=1}^{k=\infty *} \left[\frac{P_o\,\phi_k(a) + A\,\phi_k(c)}{p^2_k - \alpha^2} \right] \phi_k(c) + B_r\,\phi_r(c) = 0 \quad (23)$$

In which, for simplicity, only one intermediate support is assumed and p^1 is the kth eigenvalue of the continuous beam. The result is easily extended to any number of supports. From Eq. 21, for one inner support there is only a single μ and consequently only one coefficient

$$A = -\frac{P_o\,\phi_r(a)}{\phi_r(c)} \quad (24)$$

Substituting this value for A into Eq. 23

$$B_r = -\frac{P_o}{\phi_r(c)} \sum_{k=1}^{k=\infty *} \frac{\phi_k(a)\;\phi_k(c) - \left[\dfrac{\phi_r(a)}{\phi_r(c)}\right] \phi^2_k(c)}{p^2_k - \alpha^2} \quad (25)$$

The solution may now be written

$$(y)_p = \sum_{k=1}^{k=\infty *} B_k\,\phi_k(x) \sin \alpha\, t + B_r\,\phi_r(x) \sin \alpha\, t \quad (26)$$

Consider the simple case of the steady state vibrations of a continuous two-span beam of uniform section and subjected to the fixed pulsating load $P_o \sin \alpha\, t$ as shown in Fig. 2. The eigenfunctions for the simple beam, for example, for the beam without the intermediate support $x = c$, are

$$\phi_r(x) = \sqrt{\frac{2}{L\,m}} \sin \frac{r\,\pi\,x}{L} \quad (27)$$

and the corresponding eigenvalues are

$$p^2_r = r^4\,p^2_1 = \frac{r^4\,\pi^4\,E\,I}{L^4\,m} \quad (28)$$

From Eq. 18 the unknown coefficient

$$A = -P_o \frac{\sum_{s=1}^{s=\infty} \frac{\phi_s(a)\,\phi_s(c)}{p^2_s - \alpha^2}}{\sum_{s=1}^{s=\infty} \frac{\phi^2_s(c)}{p^2_s - \alpha^2}} = -P_o K \quad\text{.............. (29)}$$

For this particular case,

$$K = \frac{\sum_{s=1}^{s=\infty} \frac{\sin \frac{s\,\pi\,a}{L} \cdot \sin \frac{s\,\pi\,c}{L}}{s^4\,(1 - \beta^2_s)}}{\sum_{s=1}^{s=\infty} \frac{\sin^2 \frac{s\,\pi\,c}{L}}{s^4\,(1 - \beta^2_s)}} \quad\text{.................. (30)}$$

in which $\beta^2_s = \frac{\alpha^2}{s^4\,p^2_1}$. Since the integer s occurs as the fourth power in the denominators of the summations, convergence of the series is rapid, and only a few terms of the series need to be considered to obtain excellent results.

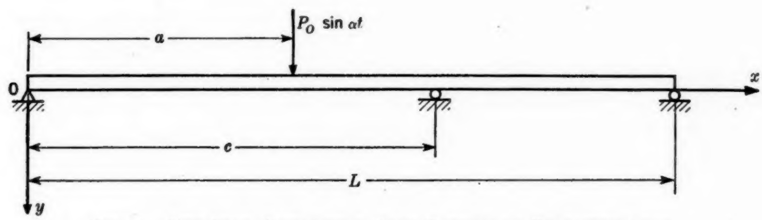

FIG. 2.—TWO-SPAN CONTINUOUS BEAM SUBJECT TO PULSATING LOAD

From Eq. 20, the steady state solution for the two-span continuous beam becomes

$$(y)_p = y_o \sum_{r=1}^{r=\infty} \frac{\sin \alpha\, t \sin \frac{r\,\pi\,x}{L}}{r^4\,(1 - \beta^2_r)} \left(\sin \frac{r\,\pi\,a}{L} - K \sin \frac{r\,\pi\,c}{L} \right) \quad\text{....... (31)}$$

in which $\beta^2_r = \frac{\alpha^2}{r^4\,p^2_1}$. The quantity

$$y_o = \frac{2\,P_o\,L^3}{\pi^4\,E\,I} \quad\text{........................ (32)}$$

is the approximate static deflection at the center of a simple supported uniform beam, of length L subjected to a concentrated load P_o at its center. It is apparent that for the simply supported beam of span length L the coefficient K is zero, and Eq. 31 reduces to the solution given by S. Timoshenko[6] and C. E. Inglis.[7]

[6] "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand Co., Inc., 2d Ed., New York, N. Y., 1937.

[7] "A Mathematical Treatise on Vibrations in Railway Bridges," by C. E. Inglis, Cambridge University Press, Cambridge, England, 1934.

Recognizing that the Fourier expansion for the concentrated pulsating load is

$$\sum_{r=1}^{r=\infty}\left(\frac{2\,P_o}{L}\times\sin\alpha\,t\times\sin\frac{r\,\pi\,a}{L}\times\sin\frac{r\,\pi\,x}{L}\right)\ldots\ldots\ldots\ldots(33)$$

and for the reaction

$$\sum_{r=1}^{r=\infty}\left(\frac{2\,K}{L}\,P_o\times\sin\alpha\,t\times\sin\frac{r\,\pi\,c}{L}\times\sin\frac{r\,\pi\,x}{L}\right)\ldots\ldots\ldots\ldots(34)$$

it can be readily shown that solution of Eq. 31 satisfies the differential Eq. 1, when $P(x,t)$ is replaced by the forces acting on the beam as well as the boundary conditions.

It is interesting to note if the frequency of the load is small in comparison with the frequencies of the simple beam, that the term β_r can be neglected, and the equation of the curve of static deflection will be obtained. The physical interpretation that can be given to K is that it is a magnitude of the reaction at $x = c$ due to a unit load at $x = a$.

Complete Solution.—The complete solution of the forced vibration problem includes both the free and steady state oscillations. For most engineering problems, the free vibrations, due to damping, gradually become insignificant and can be neglected. However, to make this work complete, the free vibrations are now considered. The method for determining the natural frequencies of the restrained system is given in the literature.[4]

The complete solution of the problem is given by

$$y = (y)_p + (y)_F = \sum_{r=1}^{r=\infty}\frac{P_o\,\phi_r(a) + \sum_{i=1}^{i=m} A_i\,\phi_r(c_i)}{p^2{}_r - \alpha^2}\sin\alpha\,t\,\phi_r(x)$$

$$+\sum_{k=1}^{k=\infty}\sum_{s=1}^{s=\infty}\frac{\sum_{i=1}^{i=m}\bar{\mu}^{(k)}{}_i\,\phi_s(c_i)}{p^2{}_s - \bar{p}^2{}_k}\,\phi_s(x)\sin(\bar{p}_k\,t + e_k)\ldots\ldots\ldots\ldots(35)$$

in which (y_p) is the particular solution given by Eq. 20. As shown by one of the authors[4] the components of the free oscillations of the continuous beam may be developed in terms of the eigenfunctions of the simple beam. Designate by $(y_k)_F$, the kth eigenfunction of the continuous beam. When expanded in terms of the eigenfunctions of the simple beam

$$(y_k)_F = \sum_{s=1}^{s=\infty}\frac{\sum_{i=1}^{i=m}\bar{\mu}^{(k)}{}_i\,\phi_s(c_i)}{p^2{}_s - \bar{p}^2{}_k}\,\phi_s(x)\sin(\bar{p}_k\,t + e_k)\ldots\ldots\ldots\ldots(36)$$

the barred symbols being used with reference to quantities associated with the free vibration of the continuous beam. The quantity $\bar{\mu}^{(k)}{}_i$ is the undetermined multiplier belonging to the ith support, when the kth mode of free vibration is considered. To determine the coefficients $\bar{\mu}^{(k)}{}_i$ and e_k the initial conditions are considered. Assume these conditions to be $y' = y = o$ at

$t = o$. From the condition $y = o$ when $t = o$, it is evident that $e_k = o$ and from $y' = o$ when $t = o$, it follows that

$$\sum_{r=1}^{r=\infty} \frac{\alpha \left[P_o \, \phi_r(a) + \sum_{i=1}^{i=m} A_i \, \phi_r(c_i) \right]}{p^2_r - \alpha^2} \phi_r(x)$$

$$+ \sum_{k=1}^{k=\infty} \sum_{s=1}^{s=\infty} \frac{\bar{p}_k \sum_{i=1}^{i=m} \bar{\mu}^{(k)}{}_i \, \phi_s(c_i)}{p^2_s - \bar{p}^2_k} \phi_s(x) = 0 \qquad (37)$$

Since this must be true for all values of x the coefficients of each $\phi_r(x)$-term in Eq. 37 must be zero. This, in turn, leads to a system of linear equations for the values of $\bar{\mu}^{(k)}$. The solution of this system of equations is required. From the form of the equations, that comprise an infinite system of equations in an infinite number of unknowns, it may be seen that the conditions are satisfied for a solution to exist. Furthermore, only a few terms need be considered because of the rapidity of the convergence. Physically this means that only a few modes of the restrained system need be considered.

For the simple two-span continuous beam under discussion Eq. 37 becomes, when $r = s$,

$$\frac{\alpha \left[P_o \, \phi_r(a) - P_o \, K \, \phi_r(c) \right]}{p^2_r - \alpha^2} + \sum_{k=1}^{k=\infty} \frac{\bar{p}_k \, \bar{\mu}^{(k)} \, \phi_r(c)}{p^2_r - \bar{p}^2_k} = 0 \qquad (38)$$

Or, in expanded form, Eq. 38 becomes

$$\frac{P_o \left[\sin \frac{r \pi a}{L} - K \sin \frac{r \pi c}{L} \right]}{r^4 (1 - \beta^2_r)} + \frac{\bar{p}_1}{\alpha} \frac{\sin \frac{r \pi c}{L}}{r^4 (1 - \bar{\omega}^2_{1r})} \bar{\mu}^{(1)}$$

$$+ \frac{\bar{p}_2}{\alpha} \frac{\sin \frac{r \pi c}{L}}{r^4 (1 - \bar{\omega}^2_{2r})} \bar{\mu}^{(2)} + \frac{\bar{p}_3}{\alpha} \frac{\sin \frac{r \pi c}{L}}{r^4 (1 - \bar{\omega}^2_{3r})} \bar{\mu}^{(3)} + \cdots = 0 \qquad (39)$$

in which $\bar{\omega}^2_{1r} = \dfrac{\bar{p}^2_1}{r^4 \, p^2_1}$; $\bar{\omega}^2_{2r} = \dfrac{\bar{p}^2_2}{r^4 \, p^2_1}$; $\cdots$. To solve for the undetermined multipliers $\bar{\mu}^{(1)}, \bar{\mu}^{(2)}, \bar{\mu}^{(3)}, \cdots$, that is the multiplier associated with each mode of free vibration, Eq. 39 is expanded for the values of $r = 1, 2, 3, \cdots$.

It can be seen that in general the solution of Eq. 37 becomes cumbersome. However, for most problems only the first few modes of free vibration need be considered, and consequently the labor for the free vibration solution is reduced considerably.

The complete solution for the forced vibrations of a two-span continuous beam subjected to a fixed pulsating load $P_o \sin \alpha t$ is then given by

$$y = y_o \sum_{r=1}^{r=\infty} \frac{\sin \alpha t \sin \frac{r \pi x}{L}}{r^4 (1 - \beta^2_r)} \left(\sin \frac{r \pi a}{L} - K \sin \frac{r \pi c}{L} \right)$$

$$+ \frac{y_o}{P_o} \sum_{k=1}^{k=\infty} \sum_{s=1}^{s=\infty} \frac{\bar{\mu}^{(k)}}{S^4 (1 - \bar{\omega}^2_{ks})} \sin \frac{s \pi c}{L} \sin \frac{s \pi x}{L} \sin \bar{p}_k t \qquad (40)$$

in which as before, $\beta^2{}_r = \dfrac{\alpha^2}{r^4\, p^2{}_1}$; $\bar{\omega}^2{}_{ks} = \dfrac{\bar{p}^2{}_k}{s^4\, p^2{}_1}$; $y_o = \dfrac{2\, P_o\, L^3}{\pi^4\, E\, I}$. Recognizing that

$$\frac{2}{L}\, \bar{\mu}^{(k)} \sin \bar{p}_k\, t \sum_{s=1}^{s=\infty} \sin \frac{s\, \pi\, c}{L} \sin \frac{s\, \pi\, x}{L} \qquad (41)$$

is the Fourier expansion of the support reaction for kth mode of free vibration, it can be readily verified that the components of free oscillation given by the second term of Eq. 40 satisfy the differential equation of motion for a continuous uniform two-span beam.

Numerical Example

As a numerical example, consider a uniform beam simply supported at the ends and supported at an interior point so that the spans are of lengths 6 and 4.5, respectively, that is $c = 6$, and $L = 10.5$ in Fig. 2. A driving force ($P_o \sin \alpha\, t$) with frequency one fourth that of the lowest natural frequency of the simple beam is applied at the center of the longer span ($a = 3$).

For this beam, the values of p[4,6] are

$$p_1 = 0.0894 \sqrt{\frac{E\, I}{m}} \qquad (42a)$$

$$\bar{p}_1 = 0.324 \sqrt{\frac{E\, I}{m}} \qquad (42b)$$

$$\bar{p}_2 = 0.636 \sqrt{\frac{E\, I}{m}} \qquad (42c)$$

$$\bar{p}_3 = 1.242 \sqrt{\frac{E\, I}{m}} \qquad (42d)$$

For $\dfrac{\alpha}{p_1} = \dfrac{1}{4}$, K is found to be 0.755. Using the first three terms of the series for the determination of the unknown multipliers, Eq. 39 becomes

$$1.165\, \bar{\mu}^{(1)} + 0.557\, \bar{\mu}^{(2)} + 0.281\, \bar{\mu}^{(3)} = 0.0509\, P_o \qquad (43a)$$

$$2.29\, \bar{\mu}^{(1)} - 0.358\, \bar{\mu}^{(2)} - 0.137\, \bar{\mu}^{(3)} = 0.0819\, P_o \qquad (43b)$$

$$0.1660\, \bar{\mu}^{(1)} + 0.734\, \bar{\mu}^{(2)} - 0.384\, \bar{\mu}^{(3)} = 0.01257\, P_o \qquad (43c)$$

Solving these equations,

$$\bar{\mu}^{(1)} = 0.0398\, P_o$$

$$\bar{\mu}^{(2)} = 0.0836\, P_o$$

$$\bar{\mu}^{(3)} = -\, 0.1500\, P_o$$

The deflection of the beam as a function of x and t is given by

$$y = y_o \left[\sin \alpha t \left(0.0509 \sin \frac{\pi x}{10.5} + 0.0819 \sin \frac{2 \pi x}{10.5} + 0.01257 \sin \frac{3 \pi x}{10.5}\right)\right.$$
$$+ \sin \bar{p}_1 t \left(- 0.00320 \sin \frac{\pi x}{10.5} - 0.00602 \sin \frac{2 \pi x}{10.5} - 0.000455 \sin \frac{3 \pi x}{10.5}\right)$$
$$+ \sin \bar{p}_2 t \left(- 0.001631 \sin \frac{\pi x}{10.5} + 0.001050 \sin \frac{2 \pi x}{10.5} - 0.00215 \sin \frac{3 \pi x}{10.5}\right)$$
$$\left. + \sin \bar{p}_3 t \left(0.000758 \sin \frac{\pi x}{10.5} - 0.000370 \sin \frac{2 \pi x}{10.5} - 0.0001035 \sin \frac{3 \pi x}{10.5}\right)\right] \quad (44)$$

The deflection at the point $x = 3$ as a function of time is given by

$$y_{x=3} = y_o \left[0.1248 \sin \alpha t - 0.00855 \sin \bar{p}_1 t - 0.001187 \sin \bar{p}_2 t + 0.000189 \sin \bar{p}_3 t\right] \ldots\ldots\ldots\ldots (45)$$

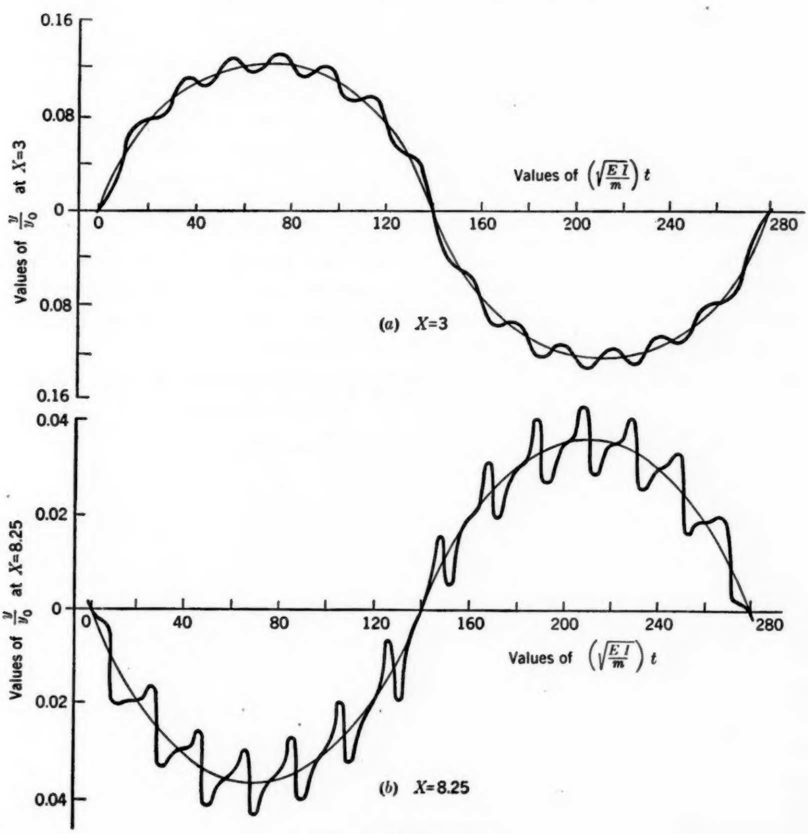

Fig. 3.—Ratio of y/y

The result is shown graphically in Fig. 3(*a*). Similarly for the point $x = 8.25$ at the center of the shorter span,

$$y_{x=8.25} = y_o\,[-\,0.0366 \sin \alpha t + 0.00345 \sin p_1 t \\ -\,0.000307 \sin p_2 t + 0.000924 \sin p_3 t] \qquad (46)$$

Fig. 3(*b*) shows a plot of the ratio of the deflection at any time to the deflection y_o. It can be seen from Fig. 3 that the deflection (shown as a wavy line) due to the free vibrations is small in comparison with the forced vibration deflection shown as a light solid line. If damping were considered, the free vibrations would disappear with increase of time. By summing the absolute values of the coefficients in Eq. 45 and Eq. 46 it is estimated that the maximum deflections at $x = 3$ and $x = 8.25$ would be $0.134\,y_o$ and $0.044\,y_o$, respectively.

Summary and Conclusion

A method has been presented for the determination of the forced oscillations of a continuous beam due to an exciting force. It has been shown that the procedure requires only a knowledge of the eigenfunctions and eigenvalues of a simple beam, that is, the beam for which all the intermediate points of support have been removed. For the simple beam the eigenfunctions and eigenvalues are either known or can be readily evaluated. The solution for the forced oscillations follows in a straightforward manner from the use of the Lagrange equations of motion. The constraints are introduced into these equations by means of undetermined multipliers. The latter are obtained as the solution of a set of simultaneous equations satisfying the constraint conditions.

Throughout this work, it is assumed that the intermediate points of support are nonyielding. It is apparent that the method can be extended to the case in which settlement occurs. Further extensions can be made to beams on flexible supports and to problems involving damping.

It may easily be seen that for certain forcing frequencies, the deflections become infinite. These correspond to the natural frequencies of vibration of the continuous beam.

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